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Phase in quantum optics

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Abstract. Dirac's prescription for quantisation does not lead to a unique phase operator for the electromagnetic field. In this paper we consider the commonly employed phase operators due to Susskind and Glogower and their extension to unitary exponential phase opertors. However, we find that phase measuring experiments respond to a different operator. We discuss the form of the measured phase operator and its properties.

1. Introduction

Dirac (1927) (see also Heitler 1954) postulated the existence of Hermitian, canonical number and phase variables in his description of the quantised electromagnetic field. Comparison with classical equations of motion led Dirac to assume that the number and phase operators obey the canonical commutation relation

$$[\hat{\phi}_D, \hat{N}] = -i \tag{1.1}$$

where the hat denotes an operator and the subscript D denotes the Dirac phase operator. This commutation relation leads to difficulties when one attempts to calculate the matrix elements of $\hat{\phi}_D$ in the representation in which $\hat{N}$ is diagonal (the photon number states) (Louisell 1963). The matrix elements between the states $\langle n'|$ and $|n\rangle$ are undefined:

$$(n-n')\langle n'|\hat{\phi}_D|n\rangle = -i\delta_{nn'}. \tag{1.2}$$

Dirac was aware that there were problems associated with his descripton of phase. However, he pointed out that the difficulties do not arise if the phase operator only appears together with the number operator in a polar decomposition of the field creation and annhilation operators (see, for example, Schweber 1984). Louisell (1963) suggested that the problem embodied in equation (1.2) could be overcome by considering periodic functions of the Dirac phase operator. (Judge and Lewis (1963) (see also Judge 1963) adopted a similar approach to the problem of angular momentum and rotation angle.) In particular, Louisell introduced the periodic operator functions $\cos\hat{\phi}_D$ and $\sin\hat{\phi}_D$ which satisfy the commutation relations

$$[\cos\hat{\phi}_D, \hat{N}] = i\sin\hat{\phi}_D \tag{1.3a}$$

$$[\sin\hat{\phi}_D, \hat{N}] = -i\cos\hat{\phi}_D. \tag{1.3b}$$

Susskind and Glogower (1964) considered a description of oscillator phase using exponential phase operators in a polar decomposition of the creation and annihilation

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operators

$$\hat{a} = \hat{e}_{\mathrm{S}}^{\mathrm{i}\phi} \hat{N}^{1/2} \tag{1.4a}$$

$$\hat{a}^{\dagger} = \hat{N}^{1/2} \hat{e}_{\mathrm{S}}^{-\mathrm{i}\phi}. \tag{1.4b}$$

The exponential phase operators of Susskind and Glogower (1964) (which we denote by subscript S) are the normalised raising and lowering operators:

$$\hat{e}_{\mathrm{S}}^{\mathrm{i}\phi} = \sum_{n=0}^{\infty} |n\rangle\langle n+1| \tag{1.5a}$$

$$\hat{e}_{\mathrm{S}}^{-\mathrm{i}\phi} = \sum_{n=0}^{\infty} |n+1\rangle\langle n|. \tag{1.5b}$$

These operators do not comute and are not unitary. Therefore, the Susskind–Glogower formalism does not allow the existence of a unique Hermitian phase operator (Susskind and Glogower 1964, Carruthers and Nieto 1968, Loudon 1973 (p 140), Lévy-Leblond 1976). The operators of equations (1.5) cannot be considered as functions of a common phase operator. It is more natural to consider the Susskind–Glogower exponential phase operators $\hat{e}_{\mathrm{S}}^{\pm\mathrm{i}\phi}$ themselves as the fundamental phase-dependent operators.

The phase operators of Susskind and Glogower have been used in discussions of the properties of coherent states (Carruthers and Nieto 1965, 1968), squeezed states (Sanders *et al* 1986) and optical amplification processes (Matthys and Jaynes 1980, Loudon and Shepherd 1984). Number-phase uncertainty relations for the Susskind–Glogower operators and number-phase minimum uncertainty states have been considered by Carruthers and Nieto (1965, 1968), Jackiw (1968), Lévy-Leblond (1976) and Sanders *et al* (1986). Phase operators have also been used in the analysis of phase measurement experiments (Gerhardt *et al* 1973, 1974, Paul 1974, Nieto 1977, Shapiro and Wagner 1984, Walker and Carroll 1984).

In this paper we reconsider the definition of a phase operator for the quantised electromagnetic field. We find that there are many suitable candidates. In particular we discuss two new candidate phase operators: a unitary exponential phase operator, $\hat{e}_{\mathrm{u}}^{\mathrm{i}\phi}$, and a cosine phase operator, $\cos_{\mathrm{M}} \phi$, corresponding to the phase-dependent property measured in homodyne (Yuen and Chan 1983 and references therein) and prepared atom (Pegg 1981) experiments. In each case we compare the new phase operators with the conventional Susskind–Glogower operators.

In § 2 we review the general properties of phase operators. Using the requirement that phase operators reproduce classical results in the suitable limits, we find the general conditions that a phase operator must satisfy. We consider the Susskind–Glogower operators and our two new phase operators as special phase operators and compare their properties.

In § 3 we consider phase measurement experiments and in § 4 we redefine the phase operator in terms of the quantities usually measured in experiments. We find that the phase operators measured in experiments are proportional to the quadrature phase operators well known from discussions of squeezing (Slusher *et al* 1985). Finally, in § 5 we compare the phase operators and discuss our results.

2. Phase operators

In this section we identify appropriate operators for the quantum mechanical description of the phase of the radiation field by exploiting the well known correspondence between a single mode of the radiation field and a simple harmonic oscillator (see, for example, Loudon 1973, p 120) and considering the Poisson bracket formulation of the classical oscillator problem. The classical action (J) and angle (ϕ) variables for a simple harmonic oscillator are related to the position and momentum by the relations (see, for example, Goldstein 1980, Carruthers and Nieto 1968)

$$q = (2J/m\omega)^{1/2} \cos\phi \tag{2.1a}$$

$$p = (2Jm\omega)^{1/2} \sin\phi \tag{2.1b}$$

where m and ω are the oscillator mass and frequency. The classical Hamiltonian is

$$\begin{aligned} H &= (p^2/2m) + (m\omega^2 q^2/2) \\ &= \omega J. \end{aligned} \tag{2.2}$$

In order to avoid the problem of multivaluedness of the phase angle it is natural to work with periodic functions of the phase ϕ. The time dependences of the phase variables $\sin\phi$ and $\cos\phi$ are given in terms of the Poisson brackets

$$(\mathrm{d}/\mathrm{d}t)\cos\phi = \{\cos\phi, H\} = \omega \sin\phi \tag{2.3a}$$

$$(\mathrm{d}/\mathrm{d}t)\sin\phi = \{\sin\phi, H\} = -\omega\cos\phi. \tag{2.3b}$$

Quantum mechanical operators whch reproduce the classical behaviour in the appropriate limit will be obtained if our operator commutators are related to the classical Poisson brackets according to the prescription (Dirac 1958)

$$[\hat{u}, \hat{v}] \leftrightarrow \mathrm{i}\hbar\{u, v\}. \tag{2.4}$$

The application of this technique to the problem of phase is due to Lerner (1968). Thus we look for Hermitian cosine and sine operators $\widehat{\cos}\,\phi$ and $\widehat{\sin}\,\phi$ obeying the commutation relations

$$[\widehat{\cos}\,\phi, \hat{N}] = \mathrm{i}\,\widehat{\sin}\,\phi \tag{2.5a}$$

$$[\widehat{\sin}\,\phi, \hat{N}] = -\mathrm{i}\,\widehat{\cos}\,\phi \tag{2.5b}$$

where the Hamiltonian operator is $\hat{H} = (\hat{N} + \frac{1}{2})\hbar\omega$. We also introduce the exponential phase operators $\hat{e}^{\pm\mathrm{i}\phi}$ by

$$\hat{e}^{\pm\mathrm{i}\phi} \equiv \widehat{\cos}\,\phi \pm \mathrm{i}\,\widehat{\sin}\,\phi \tag{2.6}$$

which from (2.5) will then obey the commutation relations

$$[\hat{e}^{\pm\mathrm{i}\phi}, \hat{N}] = \pm\hat{e}^{\pm\mathrm{i}\phi}. \tag{2.7}$$

The presumed Hermitian character of the cosine and sine operators implies that $\hat{e}^{\pm\mathrm{i}\phi}$ are the Hermitian conjugates of each other. In addition, our phase operators must reproduce the classically expected values for highly excited coherent ('classical') states:

$$\lim_{|\alpha|\to\infty} \langle\alpha|\,\widehat{\cos}\,\phi|\alpha\rangle = \cos\theta \tag{2.8a}$$

$$\lim_{|\alpha|\to\infty} \langle\alpha|\,\widehat{\sin}\,\phi|\alpha\rangle = \sin\theta \tag{2.8b}$$

where $\alpha = |\alpha|\,\mathrm{e}^{\mathrm{i}\theta}$. The Lerner criterion and classical correspondence are not sufficient, however, to define unique phase operators (Lerner 1968).

2.1. Susskind-Glogower phase operators

Dirac's (1927) original idea of radiation field phase was based upon a polar decomposition of the creation and annihilation operators into the product of an Hermitian amplitude operator and a unitary phase operator. Susskind and Glogower (1964) attempted to construct operators that were as close as possible to Dirac's conception. The resulting exponential phase operators are the bare raising and lowering operators satisfying the conditions

$$\hat{e}_S^{i\phi}|n\rangle = |n-1\rangle \tag{2.9a}$$

$$\hat{e}_S^{-i\phi}|n\rangle = |n+1\rangle. \tag{2.9b}$$

In addition, the Susskind-Glogower formalism requires the extra condition that

$$\hat{e}_S^{i\phi}|0\rangle = 0 \tag{2.10}$$

to avoid negative number states. The non-unitary character of the phase operators results from the termination of the eigenstates of $\hat{N}$ at the vacuum state $|0\rangle$. The Susskind-Glogower operators are 'one-sided unitary':

$$\hat{e}_S^{i\phi}\hat{e}_S^{-i\phi} = 1 \tag{2.11a}$$

$$\hat{e}_S^{-i\phi}\hat{e}_S^{i\phi} = 1 - |0\rangle\langle 0|. \tag{2.11b}$$

From these equations we can see that the non-commuting and non-unitary nature of $\hat{e}_S^{\pm i\phi}$ is only apparent for states of the radiation field that have a significant overlap with the vacuum

$$\langle\psi|[\hat{e}_S^{i\phi}, \hat{e}_S^{-i\phi}]|\psi\rangle = \langle\psi|0\rangle\langle 0|\psi\rangle. \tag{2.12}$$

The Susskind-Glogower phase operators obey the Lerner criterion

$$[\hat{e}_S^{\pm i\phi}, \hat{N}] = \pm\hat{e}_S^{\pm i\phi} \tag{2.13}$$

and have the required behaviour in the classical limit (equations (2.8)) (Carruthers and Nieto 1965, 1968). Extensive discussions of the Susskind-Glogower phase operators have been given by Carruthers and Nieto (1968) and Lévy-Leblond (1976).

In the quantum limit there are problems associated with interpreting the phase as described by the Susskind-Glogower operators. In particular we have

$$(\widehat{\cos}_S\,\phi)^2 + (\widehat{\sin}_S\,\phi)^2 \neq 1. \tag{2.14}$$

Also, the vacuum expectation values of $(\widehat{\cos}_S\,\phi)^2$ and $(\widehat{\sin}_S\,\phi)^2$ are $\frac{1}{4}$, not the $\frac{1}{2}$ which we would associate with a state of random phase. This implies that if it were possible to measure the Susskind-Glogower phase, a measurement of $(\widehat{\cos}_S\,\phi)^2$ would squeeze the vacuum. It should be noted that these considerations do not mean that the Susskind-Glogower formalism is inconsistent. However, it does mean that the phase operator expectation values are difficult to interpret and we require complicated uncertainty relations (Carruthers and Nieto 1965, 1968, Jackiw 1968, Lévy-Leblond 1976).

2.2. Unitary phase operators

In this subsection we introduce unitary commuting exponential phase operators (which we denote by a subscript u) $\hat{e}_u^{\pm i\phi}$. Our aim is to realise Dirac's idea of a polar decomposition of the creation and annihilation operators into Hermitian and unitary parts. We achieve our objective by extending the normal harmonic oscillator Hilbert space to include negative number states. We note that the derivation of the harmonic oscillator eigenstates merely requires a ground state that is annihilated by the annihilation operator so that the energy spectrum is bounded from below (see, for example, Merzbacher 1970). Negative energy states are not precluded, but they must be decoupled from the positive energy ground state that is annihilated by the annihilation operator. We shall see that the states containing a negative number of photons are inaccessible to a physical system so their mere existence in the formalism does not predict any new phenomena in quantum electrodynamics.

We define the unitary exponential phase operators as the normalised raising and lowering operators extending over the complete Hilbert space of all positive and negative photon number states:

$$\hat{e}_{\mathrm{u}}^{\mathrm{i}\phi} = \sum_{n=-\infty}^{\infty} |n\rangle\langle n+1| \tag{2.15a}$$

$$\hat{e}_{\mathrm{u}}^{-\mathrm{i}\phi} = \sum_{n=-\infty}^{\infty} |n+1\rangle\langle n|. \tag{2.15b}$$

With this extended basis of orthonormal oscillator states, the resolution for the identity becomes

$$\sum_{n=-\infty}^{\infty} |n\rangle\langle n| = 1. \tag{2.16}$$

The unitarity of $\hat{e}_{\mathrm{u}}^{\pm\mathrm{i}\phi}$ results from the absence of a cutoff in the summations defining the operators.

We retain the property that the number states are eigenstates of the number operator $\hat{N}$ with eigenvalues n

$$\hat{N}|n\rangle = n|n\rangle \tag{2.17}$$

for all positive and negative n. The polar decomposition of the creation and annihilation operators into an Hermitian amplitude and unitary phase then requires that

$$\hat{a} = \hat{e}_{\mathrm{u}}^{\mathrm{i}\phi}|\hat{N}^{1/2}| \tag{2.18a}$$

$$\hat{a}^{\dagger} = |\hat{N}^{1/2}|\hat{e}_{\mathrm{u}}^{-\mathrm{i}\phi} \tag{2.18b}$$

where $|\hat{N}^{1/2}|$ is the Hermitian amplitude operator

$$|\hat{N}^{1/2}| \equiv \sum_{n=-\infty}^{\infty} |n^{1/2}||n\rangle\langle n|. \tag{2.19}$$

The expectation value of the commutator of $\hat{a}$ and $\hat{a}^{\dagger}$ depends upon whether the system is in a positive or negative photon number state:

$$[\hat{a}, \hat{a}^{\dagger}] = \sum_{n=0}^{\infty} |n\rangle\langle n| - \sum_{n=-\infty}^{-1} |n\rangle\langle n|. \tag{2.20}$$

For the creation and annihilation operators to correspond to the negative and positive frequency components of the free field, we require the free-field Hamiltonian to be

$$\hat{H} = (\hat{N} + \tfrac{1}{2})\hbar\omega \tag{2.21}$$

with both positive and negative energy eigenvalues.

The destruction of the vacuum state $|0\rangle$ by the annihilation operator and the destruction of the state $|-1\rangle$ by the creation operator are due to the Hermitian amplitude parts of the operators (2.18). Physical couplings to the radiation field take place via the creation and annihilation operators, the fundamental coupling being of the form

$$\hat{H}_I = (c \text{ numbers})(\hat{a} + \hat{a}^\dagger)(\text{current operators}). \tag{2.22}$$

It follows that the negative energy states cannot be coupled to any physical, i.e. positive energy, states because the Hamiltonian matrix elements between positive and negative energy states are always zero:

$$\langle n|\hat{H}_I|n'\rangle = 0 \tag{2.23}$$

if $|n\rangle$ is a negative energy state and $|n'\rangle$ is a positive energy state. Therefore, a field initially in a superposition of positive energy eigenstates can never evolve into a state containing a negative energy eigenstate. This lack of coupling between the positive and negative energy states means that quantum electrodynamical systems are restricted to the positive energy subspace where the familiar relations

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{2.24}$$

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = 1 \tag{2.25}$$

are true.

The unitary phase operators obey the Lerner criterion if we employ the extended basis number and Hamiltonian operators

$$[\hat{e}_{\mathrm{u}}^{\pm \mathrm{i}\phi}, \hat{N}] = \pm \hat{e}_{\mathrm{u}}^{\pm \mathrm{i}\phi}. \tag{2.26}$$

The similarity between the unitary and the Susskind–Glogower phase operators ensures that the unitary phase operators have the required behaviour in the classical (positive energy) limit.

As with the Susskind–Glogower operators, there are problems associated with the unitary phase operators. In the unitary formalism the cosine and sine phase operators obey the trigonometric identity

$$(\widehat{\cos}_{\mathrm{u}}\, \phi)^2 + (\widehat{\sin}_{\mathrm{u}}\, \phi)^2 = 1 \tag{2.27}$$

and the vacuum state expectation values of $(\widehat{\cos}_{\mathrm{u}}\, \phi)^2$ and $(\widehat{\sin}_{\mathrm{u}}\, \phi)^2$ are $\frac{1}{2}$. However, these properties rely on the existence of unphysical negative photon number states. The unitary phase operators are unmeasurable because the fundamental interaction Hamiltonian (equation (2.22)) does not couple positive and negative number states together. Therefore no measuring device can be constructed that is sensitive to the negative number states in the definition of the unitary phase operators.

In addition, the Susskind–Glogower phase operators do not correspond to the quantities measured in homodyne (Yuen and Shapiro 1980) and prepared atom (Pegg 1981) experiments. We now turn our attention to defining different phase operators that correspond to these usual phase measuring experiments.

2.3. *Measured phase operators*

In the previous subsection we noted that the fundamental radiation–matter coupling in quantum electrodynamics is via the field creation and annihilation operators.

Therefore it seems natural to consider the creation and annihilation operators as fundamental and to construct phase operators in terms of $\hat{a}$ and $\hat{a}^\dagger$. We noted previously that the Lerner criterion does not define a unique phase operator; indeed the creation and annihilation operators themselves obey the Lerner criterion

$$[\hat{a}, \hat{N}] = \hat{a} \tag{2.28a}$$

$$[\hat{a}^\dagger, \hat{N}] = -\hat{a}^\dagger. \tag{2.28b}$$

In fact we can construct any operators of the general form

$$\hat{e}^{i\phi} = \hat{a}f(\hat{N}) + g(\hat{N})\hat{a} \tag{2.29a}$$

$$\hat{e}^{-i\phi} = f(\hat{N})\hat{a}^\dagger + \hat{a}^\dagger g(\hat{N}) \tag{2.29b}$$

where f and g are well behaved functions, which also satisfy the Lerner criterion. The Susskind-Glogower expressions (equations (1.4)) are just one particular member of this larger set.

The choice of phase operators is further constrained by the condition that they must reproduce the classically expected values for highly excited coherent states. Therefore the creation and annihilation operators themselves are not suitable as phase operators, although suitable phase operators of the form presented in equations (2.29) can be constructed. In particular Lerner (1968) has advocated the use of the symmerical expressions, either

$$\hat{e}^{i\phi} = \tfrac{1}{2}[(\hat{N}+\tfrac{1}{2})^{-1/2}\hat{a} + \hat{a}(\hat{N}+\tfrac{1}{2})^{-1/2}] \tag{2.30a}$$

$$\hat{e}^{-i\phi} = \tfrac{1}{2}[(\hat{N}+\tfrac{1}{2})^{-1/2}\hat{a}^\dagger + \hat{a}^\dagger(\hat{N}+\tfrac{1}{2})^{-1/2}] \tag{2.30b}$$

or

$$\hat{a} = \tfrac{1}{2}[(\hat{N}+\tfrac{1}{2})^{1/2}\hat{e}^{i\phi} + \hat{e}^{i\phi}(\hat{N}+\tfrac{1}{2})^{1/2}] \tag{2.31a}$$

$$\hat{a}^\dagger = \tfrac{1}{2}[(\hat{N}+\tfrac{1}{2})^{1/2}\hat{e}^{i\phi} + \hat{e}^{i\phi}(\hat{N}+\tfrac{1}{2})^{1/2}]. \tag{2.31b}$$

In this subsection we construct phase operators that correspond to the usual operational definition of a phase measurement. In § 3 we shall see that the quantity suggested by homodyne (Yuen and Shapiro 1980) and prepared atom (Pegg 1981) experiments is

$$\widehat{\cos}_M\, \phi = k(\hat{a} + \hat{a}^\dagger) \tag{2.32}$$

where k is a state-dependent c number (obtained by means of an independent experiment). We use the subscript M to denote these measured phase operators. The number k must be chosen so that

$$\lim_{|\alpha|\to\infty} \langle\alpha|k(\hat{a} + \hat{a}^\dagger)|\alpha\rangle = \cos\theta \tag{2.33}$$

where $\alpha = |\alpha|\, e^{i\theta}$. We also define $\widehat{\sin}_M\, \phi$ to be

$$\widehat{\sin}_M\, \phi = -ik(\hat{a} - \hat{a}^\dagger). \tag{2.34}$$

We give an analysis of the phase measurements that lead us to define our measured phase operators in § 3.

Our choice, for reasons discussed later, is to define

$$\widehat{\cos}_M\, \phi = \frac{\hat{a} + \hat{a}^\dagger}{2(\bar{n}+\tfrac{1}{2})^{1/2}} \tag{2.35a}$$

$$\widehat{\sin}_M\, \phi = \frac{\hat{a} - \hat{a}^\dagger}{2i(\bar{n}+\tfrac{1}{2})^{1/2}} \tag{2.35b}$$

where $\bar{n}$ is the mean photon number of the measured field. This choice is arrived at by placing the energy or intensity in the denominator of the classical phase measurement experiments. Here, in contrast to the operators of Lerner (1968, equations (2.30)), the energy denominator takes the form of an independently derived c number rather than an operator. The phase operators exhibit the classically expected property that

$$\langle(\widehat{\cos}_{\mathrm{M}}\,\phi)^2\rangle+\langle(\widehat{\sin}_{\mathrm{M}}\,\phi)^2\rangle=1. \tag{2.36}$$

The major problem associated with these phase operators is that their spectra are not bounded by the interval $(-1, 1)$. This may be demonstrated by considering the expectation values of the operators $(\cos_{\mathrm{M}}\,\phi)^m$. We address this problem in § 4. However, as these operators correspond to the quantities measured in conventional phase measurements it seems natural to adopt them as phase operators.

In table 1 we list some of the properties of the Susskind-Glogower, unitary and measured phase operators in order to highlight the differences between them.

Table 1. Some of the properties of the phase operators discussed in this paper.

	Susskind-Glogower	Unitary	Measured
$\hat{e}^{i\phi}$	$\sum_{n=0}^{\infty} \|n\rangle\langle n+1\|$	$\sum_{n=-\infty}^{\infty} \|n\rangle\langle n+1\|$	$(\bar{n}+\frac{1}{2})^{-1/2}\hat{a}$
$\hat{e}^{-i\phi}$	$\sum_{n=0}^{\infty} \|n+1\rangle\langle n\|$	$\sum_{n=-\infty}^{\infty} \|n+1\rangle\langle n\|$	$(\bar{n}+\frac{1}{2})^{-1/2}\hat{a}^{\dagger}$
$[\widehat{\cos}\,\phi, \widehat{\sin}\,\phi]$	$\frac{1}{2}i\|0\rangle\langle 0\|$	0	$\frac{1}{2}i(\bar{n}+\frac{1}{2})^{-1}$
$\langle 0\|(\widehat{\cos}\,\phi)^2\|0\rangle$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$
$\langle 0\|(\widehat{\sin}\,\phi)^2\|0\rangle$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$
$\langle 0\|(\widehat{\cos}\,\phi)^4\|0\rangle$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{4}$
$\langle 0\|(\widehat{\sin}\,\phi)^4\|0\rangle$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{4}$
$\langle(\widehat{\cos}\,\phi)^2\rangle+\langle(\widehat{\sin}\,\phi)^2\rangle$	$1-\frac{1}{2}\langle\|0\rangle\langle 0\|\rangle$	1	1

3. Phase measurements

Classically an absolute phase has no meaning and all measurements must be made relative to the phase of the reference system. The same can be expected for quantum mechnical systems, with the observable quantity being the phase difference between the quantum system and a reference oscillator. If the reference oscillator is in a highly excited coherent state then it has a well defined phase (equations (2.8)). In what follows we choose this reference phase to be precisley zero.

An observable is measured by means of an effect on the measuring apparatus. Electrodynamic fields interact with matter by means of an interaction Hamiltonian which, in the quantum mechanical case, will involve the creation and annihilation operators $\hat{a}^{\dagger}$ and $\hat{a}$. It is the properties of these operators which will predict the outcome of such measurements. We discuss briefly two physical processes where it is known that the phase of a classical field has a measurable effect on the measuring apparatus. We postulate that the corresponding effect produced by a quantum field will also be a phase effect, from which an operational means of defining the phase can be determined. A boundary condition which must be satisfied is that, when the measured field is also in a coherent state, with mean photon number $\bar{n}$, the result of a phase measurement must tend to the classically expected value as $\bar{n}$ increases.

The first measurement process involves homodyne detection (see, for example, Yuen and Shapiro 1980, Yuen and Chan 1983), i.e. mixing two fields of the same frequency and measuring the total intensity. Here the reference field will be a coherent field, whose intensity we shall allow to tend to infinity, and whose phase is defined to be zero. Classically this corresponds to a reference field which in the dipole approximation at the detector is $E_R \cos \omega t$. If this is mixed with a classical field $E_M \cos(\omega t + \phi)$ with a fluctuating phase ϕ, it is not difficult to show that for $I_R \gg I_M$, where I_R and I_M are the cycle-averaged intensities of the reference and measured fields

$$\cos \phi = \frac{I - I_R}{2(I_R I_M)^{1/2}}. \tag{3.1}$$

Here I is the total cycle-averaged intensity measured by the detector. Clearly the measurement needs to be made in a short time compared with the characteristic time of the fluctuations.

In the quantum mechanical case, where intensities are again measured, for example by photoelectron counts, a measurement of $\cos \phi$ could be defined in terms of quantities associated with those on the right-hand side of (3.1). By using suitable beam splitting and path differences a single measurement of $I - I_R$ could be made (Yuen and Shapiro 1980, Yuen and Chan 1983). Alternatively, because I_R is assumed to be without fluctuations, and thus the same at all times, a separate measurement of I_R could be made. It would be difficult in practice to measure $(I_M)^{1/2}$ at precisely the same time and place as the measurement of $I - I_R$ is performed. Also, in usual experiments a measurement of $(I_M)^{1/2}$ is not performed, either immediately before the measurement of $I - I_R$ or afterwards. Consequently, finding the operator counterpart of (3.1) involves replacing only the numerator with an operator $\hat{I} - \hat{I}_R$; the denominator will be a c number chosen to give correct dimensions and the correct limiting behaviour. The resulting operator will act on both the reference and measured fields. In order to obtain an operator which acts only on the measured field we use $\langle\beta|\hat{I} - \hat{I}_R|\beta\rangle$ in the numerator, where $|\beta\rangle$ is the state of the strong coherent reference field. This latter field is chosen to have phase zero and the eigenvalue β of its annihilation operator $\hat{b}$ is real. Writing $\hat{I}$ in terms of the combined field operators $\hat{b} + \hat{a}$ and its Hermitian conjugate, $\hat{I}_R$ in terms of the reference field operators $\hat{b}$ and $\hat{b}^{\dagger}$ and letting $\beta \to \infty$, we find the phase operator corresponding to the operation of homodyne measurement to be simply

$$\widehat{\cos}_M \phi = k(\hat{a} + \hat{a}^{\dagger}). \tag{3.2}$$

This operator is a normalised quadrature phase operator. The reduced fluctuations in one of the quadrature phases associated with squeezed states (Walls 1983 and references therein) imply reduced noise in the measured phase operators. We have used the fact that the c number representing the denominator in (3.1) must be proportional to β as $\beta \to \infty$ in order to obtain a finite expression, as one would with a classical reference field. Thus k is a c number which depends only on the measured field and has the dimensions of the inverse square root of a photon number. Also, if the measured field is in a coherent state, with mean photon number $\bar{n}$, k must approach $\frac{1}{2}(\bar{n})^{-1/2}$ for large

$\bar{n}$ in order to obtain the correct classical limit. Without loss of generality we can write

$$k = \tfrac{1}{2}(\bar{n} + F)^{-1/2} \tag{3.3}$$

where F is to be determined or defined subject to the condition that $F \ll \bar{n}$ for large $\bar{n}$.

The second method for the measurement of a phase associated property involves the interaction of the field with a two-level atom in a particular superposition state. It is well known that the occurrence of absorption or stimulated emission depends on the relative phases of the field and the prepared atomic state. It should be possible, therefore, to find a measure of the phase of the field by examining the initial change of the atomic state at the instant of interaction with the field. The atomic state could be prepared by a $\pi/2$ pulse from a reference field at an earlier time, in a similar manner to that described by Pegg (1981). To be specific, consider a two-level atom with excited and ground states $|e\rangle$ and $|g\rangle$ with transition frequency ω resonant with both the reference and measured field frequencies. The prior action of the intense reference field is equivalent to that of a classical field (with zero phase) so the Hamiltonian is

$$H = \omega|e\rangle\langle e| + \lambda E_{\mathrm{R}}(t) \cos \omega t(|e\rangle\langle g| + |g\rangle\langle e|) \tag{3.4}$$

where λ is the coupling constant. It is convenient to work in an interaction picture. We use the unitary operator

$$T = \exp(\mathrm{i}|e\rangle\langle e|\omega t) \tag{3.5}$$

to transform to a reference frame in which the field $E_{\mathrm{R}}(t) \cos \omega t$ becomes an effective field $\frac{1}{2}E_{\mathrm{R}}(t)$ when the rotating-wave approximation is made (see, for example, Knight and Allen 1983). In this frame the Hamiltonian is time independent:

$$H = \tfrac{1}{2}\lambda E_{\mathrm{R}}(t)(|e\rangle\langle g| + |g\rangle\langle e|). \tag{3.6}$$

The action of a $\pi/2$ pulse is to put an initially ground-state atom into a coherent superposition state which, in this frame, is

$$|A\rangle = 2^{-1/2}(|g\rangle - \mathrm{i}|e\rangle). \tag{3.7}$$

A more detailed discussion of the action of classical pulses on two-level atoms is given by Allen and Eberly (1975). The prepared atom retains the well defined phase information of the reference field in its dipole moment.

If a general fluctuating classical field $E_{\mathrm{M}}(t) \cos(\omega t - \phi(t))$ is applied to the prepared atom at time t_1 the value of $\langle\sigma_z\rangle$ will change, where $\sigma_z \equiv |e\rangle\langle e| - |g\rangle\langle g|$, in a manner dependent on the phase of the field. In the interaction picture in the rotating-wave approximation the Hamiltonian is

$$H(t) = \tfrac{1}{2}\lambda E_{\mathrm{M}}(t)(\mathrm{e}^{\mathrm{i}\phi(t)}|e\rangle\langle g| + \mathrm{e}^{-\mathrm{i}\phi(t)}|g\rangle\langle e|). \tag{3.8}$$

Under the action of this second field the initial rate of change of $\langle\sigma_z\rangle$ for the atom in the prepared atomic state is easily found from $\mathrm{i}\langle A|[H, \sigma_z](t_1)|A\rangle$ to be proportional to $\cos \phi(t_1)$, i.e.

$$\langle\dot{\sigma}_z(t_1)\rangle = \tfrac{1}{2}\lambda E_{\mathrm{M}}(t_1) \cos \phi(t_1). \tag{3.9}$$

In the quantum mechanical case the prepared atom is exposed at time t_1 to a quantised electric field. As in the semiclassical treatment above we expect the value of $\langle\sigma_z\rangle$ to change in a manner dependent on the phase of the field. In the interaction picture, and making the rotating-wave approximation, the Hamiltonian is

$$H = \tfrac{1}{2}\Lambda(\hat{a}|e\rangle\langle g| + \hat{a}^{+}|g\rangle\langle e|) \tag{3.10}$$

where Λ is the fully quantum mechanical coupling constant. Under the action of the quantised field the initial rate of change of $\langle\sigma_z\rangle$ in the prepared atomic state is found from $\mathrm{i}\langle f|\langle A|[H,\sigma_z]|A\rangle|f\rangle$, where $|f\rangle$ is the field state at t_1, to be

$$\langle\dot{\sigma}_z(t_1)\rangle = \tfrac{1}{2}\Lambda\langle f|\hat{a}+\hat{a}^{\dagger}|f\rangle \tag{3.11}$$

which is proportional to $\langle\widehat{\cos}_{\mathrm{M}}\,\phi\rangle$. For a field in a strong coherent state with complex amplitude $\alpha=|\alpha|\,\mathrm{e}^{\mathrm{i}\theta}$ it is clear that $\langle\dot{\sigma}_z(t_1)\rangle$ is proportional to $\cos\theta$. For a general field, comparison of (3.9) and (3.11) shows that, as with the homodyne measurement, the phase operator corresponding to the operation of prepared atom phase measurement is $\widehat{\cos}_{\mathrm{M}}\,\phi$.

The corresponding sine phase operator, $\widehat{\sin}_{\mathrm{M}}\,\phi$, can be measured by altering the phase of the reference oscillator.

4. Choice of measured phase operators

From the preceding work it is clear that a definition of $\widehat{\cos}_{\mathrm{M}}\,\phi$ as

$$\widehat{\cos}_{\mathrm{M}}\,\phi = k(\hat{a}+\hat{a}^{\dagger}) \tag{4.1}$$

is at least in accord with measurable phase-dependent properties. Indeed, the original phase measurements of Gerhardt *et al* (1973, 1974) involved a homdyne technique. In this, the experiments appear to be measurements of the observable associated with $\widehat{\cos}_{\mathrm{M}}\,\phi$ rather than measurements of the Susskind-Glogower (1964) phase $\widehat{\cos}_{\mathrm{S}}\,\phi$ that they had intended to measure (Gerhardt *et al* 1974, Nieto 1977, Lévy-Leblond 1977). In classical homodyne experiments the maximum attainable value for the measured phase is $\cos\phi=1$. However, in a quantum homodyne experiment the spectrum of $\widehat{\cos}_{\mathrm{M}}\,\phi$ is unbounded. This may be demonstrated by considering the expectation value of $(\widehat{\cos}_{\mathrm{M}}\,\phi)^{2m}$. If the spectrum of $(\widehat{\cos}_{\mathrm{M}}\,\phi)^{2m}$ is bounded then the spectrum of $(\widehat{\cos}_{\mathrm{M}}\,\phi)^{2m}$ will also be bounded and its expectation value will be finite for all m. This proves not to be the case. We calculate expectation values of the operator $(\widehat{\cos}_{\mathrm{M}}\,\phi)^{2m}$ by using the generating function $\hat{G}(x)$

$$\hat{G}(x) = \exp(x\,\widehat{\cos}_{\mathrm{M}}\,\phi). \tag{4.2}$$

We can write the generating function in a normally ordered form by using the Baker-Campbell-Hausdorf theorem (Louisell 1973, Hong and Mandel 1985)

$$\hat{G}(x) = \exp(x^2k^2/2):\hat{G}(x): \tag{4.3}$$

where the colons denote normal ordering. Expanding both sides of (4.3) as a series and equating coefficients of $x^{2m}/(2m)!$ we find a series for $(\widehat{\cos}_{\mathrm{M}}\,\phi)^{2m}$ in terms of normally ordered operators

$$\begin{aligned}(\widehat{\cos}_{\mathrm{M}}\,\phi)^{2m} = {}&:(\widehat{\cos}_{\mathrm{M}}\,\phi)^{2m}: + \tfrac{1}{2}k^2 2m(2m-1)/1!:(\widehat{\cos}_{\mathrm{M}}\,\phi)^{2m-2}:\\ &+(\tfrac{1}{2}k^2)^2 2m(2m-1)(2m-2)(2m-3)/2!:(\widehat{\cos}_{\mathrm{M}}\,\phi)^{2m-4}:+\ldots\\ &+(\tfrac{1}{2}k^2)^m(2m)!/m!.\end{aligned} \tag{4.4}$$

This series involves terms containing normally ordered even powers of $\widehat{\cos}_{\mathrm{M}}\,\phi$ with positive coefficients and a constant term $(k^2/2)^m(2m)!/m!$ which diverges as m increases. By choosing m large enough, we can make $\langle(\widehat{\cos}_{\mathrm{M}}\,\phi)^{2m}\rangle$ greater than any number we choose. This implies that the spectrum of $(\widehat{\cos}_{\mathrm{M}}\,\phi)^{2m}$, and therefore the spectrum of $\widehat{\cos}_{\rm M}\,\phi$, is unbounded. In a classical homodyne experiment, the largest realisation of the intensity of the combined reference and signal fields can be chosen to correspond approximately to $\cos\phi = 1$. In this way the apparatus may be calibrated. However, the spectrum of $\widehat{\cos}_{\rm M}\,\phi$ is unbounded and no such technique can be applied to calibrate a quantum phase measurement. There is no well defined method for using the largest realisation of the measured intensity to determine k

It then remains to choose a suitable value for k in the definition of $\widehat{\cos}_{\rm M}\,\phi$. From the above discussion we cannot choose a value of k other than zero such that all experimentally measured values of $\widehat{\cos}_{\rm M}\,\phi$ are less than one. However, guided by the relationship between k and $\frac{1}{2}(I_{\rm M})^{-1/2}$ in the classical expression for $\cos\phi$ (equation (3.1)) and remembering that c numbers can be obtained in quantum mechanics from expectation values, a reasonable choice for k is $\frac{1}{2}(\bar{n}+\frac{1}{2})^{-1/2}$. This can be compared with the operator expressions of Susskind and Glogower (1964) where, for example the operator $(\hat{N}+1)^{-1/2}$ is on the left of $\hat{a}$ or $\hat{N}^{-1/2}$ is on the right of $\hat{a}$ in

$$\hat{e}^{i\phi}_{\rm S} = (\hat{N}+1)^{-1/2}\hat{a} = \hat{a}\hat{N}^{-1/2} \tag{4.5}$$

and the symmetrical expressions of Lerner (equations (2.30) and (2.31)). In the definition of $\widehat{\cos}_{\rm M}\,\phi$, because $\bar{n}$ commutes with $\hat{a}$, the corresponding expressions have $(\bar{n}+\frac{1}{2})^{-1/2}$ on the right or left of $\hat{a}$. Our operator definition is thus

$$\widehat{\cos}_{\rm M}\,\phi = (\hat{a}+\hat{a}^{\dagger})/2(\bar{n}+\tfrac{1}{2})^{1/2}. \tag{4.6}$$

By altering the phase of the reference field by $\pi/2$ another measurement can be performed which, in the classical limit, behaves as $\sin\phi$. Following the same procedure as above gives the corresponding definition of $\widehat{\sin}_{\rm M}\,\phi$ as

$$\widehat{\sin}_{\rm M}\,\phi = (\hat{a}-\hat{a}^{\dagger})/2{\rm i}(\bar{n}+\tfrac{1}{2})^{1/2}. \tag{4.7}$$

It is apparent that the phase measurements of Gerhardt *et al* (1973, 1974) and, for example, Walker and Carroll (1984) correspond more closely to $\widehat{\cos}_{\rm M}\,\phi$ and $\widehat{\sin}_{\rm M}\,\phi$ than to $\widehat{\cos}_{\rm S}\,\phi$ and $\widehat{\sin}_{\rm S}\,\phi$. In the experiments of Gerhardt *et al* the phase measurements were normalised by using the largest occurring intensity. Thus the normalisation constant k will only be approximately the same as that above.

5. Discussion

In this paper we have investigated appropriate operators for the quantum mechanical description of the phase of the radiation field. In particular we have considered the Susskind–Glogower, unitary and measured phase operators. All the phase operators exhibit non-classical behaviour for quantum states but reproduce classical phase properties in the appropriate limit.

The original motivation for the introduction of phase operators was to describe the electric field in terms of polar (amplitude and phase) variables. Dirac (1927) suggested that the single mode creation and annihilation operators could be factorised into Hermitian amplitude and unitary exponential phase operators. Susskind and Glogower (1964) demonstrated that the exponential phase operators obtained from such a factorisation are only one-sided unitary. Of the number of different possible exponential phase operators which satisfy the Lerner criterion, and which involve only positive energy states, the Susskind–Glogower operator is that which is closest to being unitary. A nearly unitary operator is, nevertheless, still non-unitary. If the region of interest is that in which the non-unitarity is most apparent, this near unitarity is not a great advantage. In particular, fields with very small mean photon numbers, where non-classical effects might be expected to be observable, represent such a region. If a unitary operator is required, we see no alternative to using the complete Hilbert space which includes the negative energy states, but this requires an infinite set of unmeasurable states which are inaccessible for any physical system. On the other hand, if unitarity is not a necessary requirement, we prefer to abandon it entirely and to define phase operators in terms of the quantities actually measured in usual phase measuring experiments. This leads us to consider the single mode creation and annihilation operators ($\hat{a}^\dagger$ and $\hat{a}$) as the fundamental field operators and to define measured quantities in terms of them. The analysis presented in § 3 of this paper demonstrated that usual phase measuring experiments correspond to measurements of operators proportional to $\hat{a}+\hat{a}^\dagger$. Therefore we define our measured cosine phase operator to be proportional to $\hat{a}+\hat{a}^\dagger$. We note that in squeezing experiments it is in fact the fluctuations in the measured cosine and sine phase operators which can be said to be squeezed (Walls 1983 and references therein). Indeed, we can equally well discuss squeezed states in terms of the measured phase operators or the usual quadrature phases which are identical apart from a normalisation factor.

In conclusion we suggest the adoption of $\widehat{\cos}_{\mathrm{M}}\,\phi$ and $\widehat{\sin}_{\mathrm{M}}\,\phi$ as an operational definition of phase measurement.

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